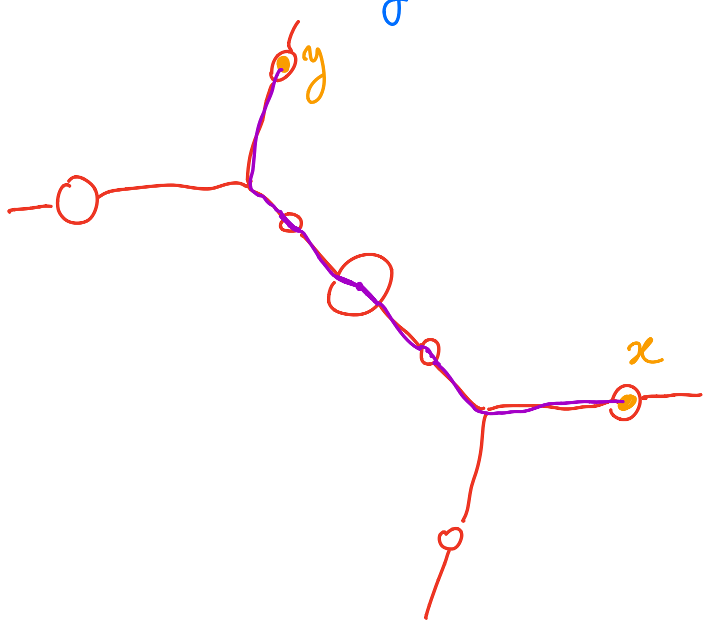
Core Entropy of Quadratic Polynomials

Let f be a quad. polynomial with connected & locally connected Julia set. In particular, it is also path connected.

Def.: Given $x, y \in J(f)$, let $[x, y]$ be the regulated arc between x and y.

That is, $\gamma = [x, y]$ is an arc joining x, y, which lies inside the filled Julia set and s.t., for any Fatou component $\mathcal{U}$, $\gamma \cap \mathcal{U}$ is contained in the union of two internal rays.



Def.: The Hubbard tree is the set

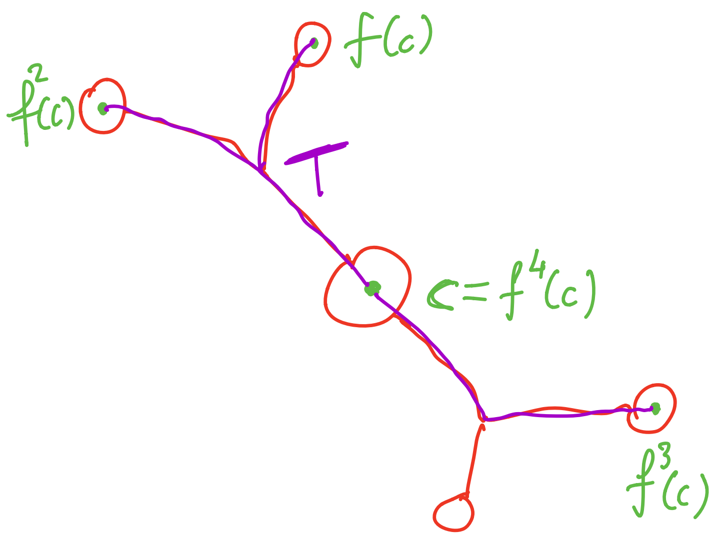
$$T = \bigcup_{i,j \geq 0} [f^i(c), f^j(c)]$$

where c is the critical point.

E.g.: f is postcritically finite,

$$\mathcal{P} = \{c, f(c), \ldots, f^{p-1}(c)\}$$

is finite. Then you consider the union of all arcs between points of $\mathcal{P}$.

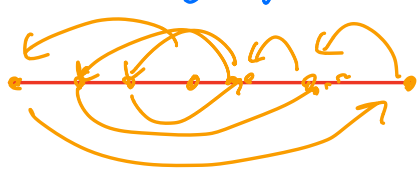


Lemma $f(T) \subseteq T$

but in general $f^{-1}(T) \neq T$.

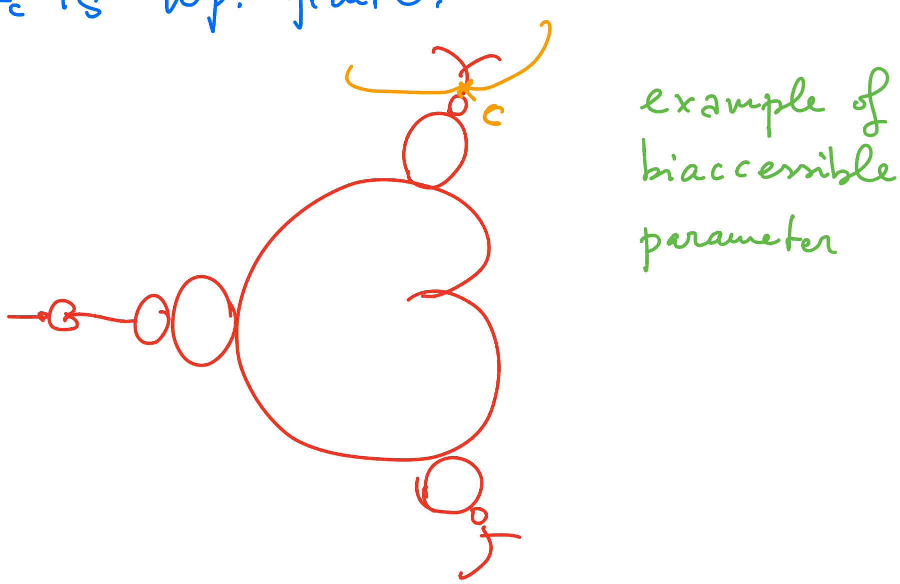
Def.: f is *topologically finite* if $J(f)$ is connected & locally connected and the Hubbard tree has finitely many ends.

E.g.: ① Every real quadratic polynomial is topologically finite



② Every postcritically finite quad. poly is top. finite.

③ If c is biaccessible (i.e. there are 2 rays landing there) in the Mandelbrot set, then f_c is top. finite.



Def: The core entropy of a top. finite polynomial f is

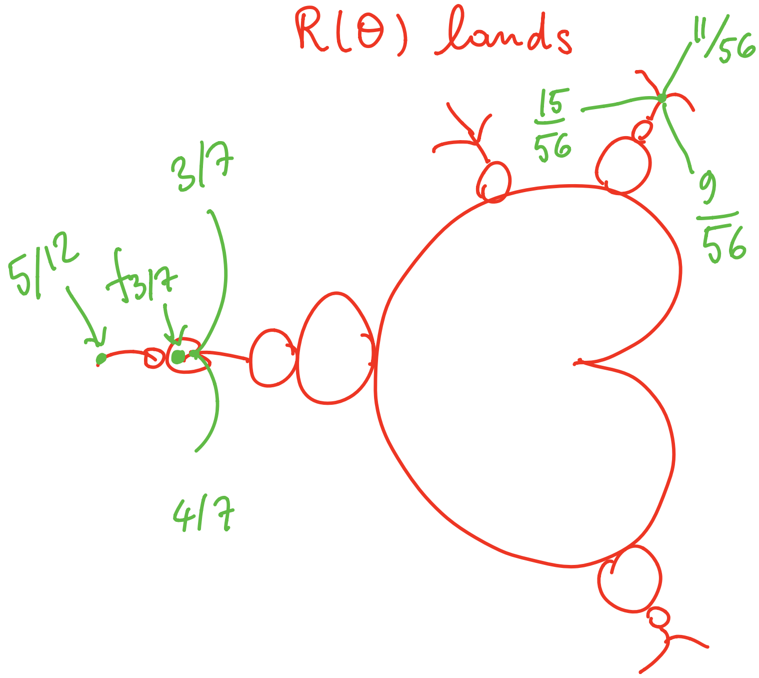
$$h(f) := h_{top}\left(f|_T : T \to T\right)$$

External angle parameterization

$$\mathbb{Q}/\mathbb{Z} \ni \theta \longmapsto f_\theta \text{ a PCF polynomial} \longmapsto h(\theta) = h(f_\theta)$$

if $\theta = \frac{p}{q}$, q odd then f_θ is center of hyp. cpt at root of which $R(\theta)$ lands

if $\theta = \frac{p}{q}$, q even then $R(\theta)$ lands at a Misiurewicz, which we take as f_θ.

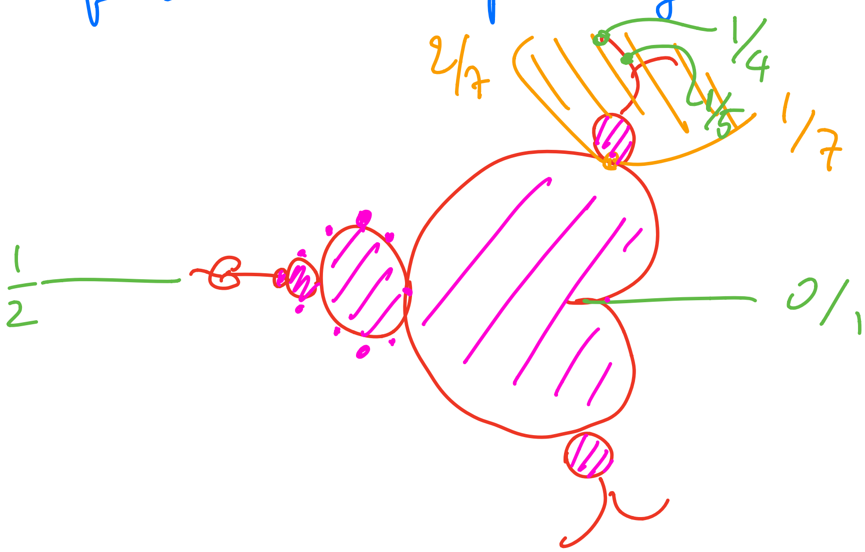


$\boxed{Q}$ ① How to compute $h(\theta)$?

② How does h vary on parameter space? Does it tell us anything about it?

<u>Conjecture</u> (T. - proven by Dudko-Schleicher)

If $R(\theta_1)$, $R(\theta_2)$ land at the same point on $\mathcal{M}$, the entropy on $[\theta_1, \theta_2]$ is maximized at its pseudocenter, i.e. the dyadic rational inside $[\theta_1, \theta_2]$ of lowest complexity.



$\text{pseudo}\left(\left[\frac{1}{7}, \frac{2}{7}\right]\right)$

<u>Note</u> each interval has a <u>unique</u> pseudocenter.

Def.: The main molecule is the set of hyperbolic components which have a finite chain of adjacent components connecting them to the main cardioid.

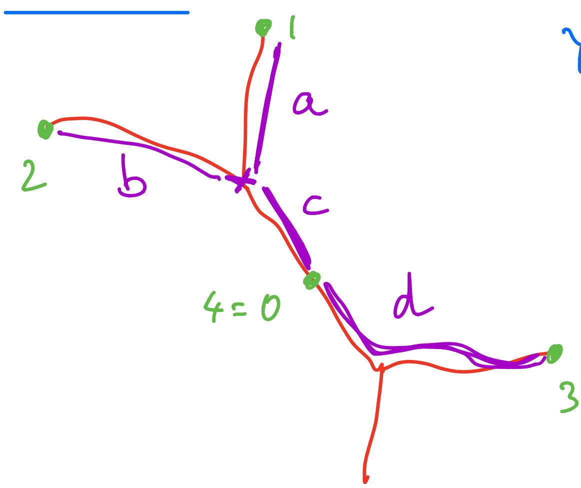
Thm The zero set of the core entropy function is the closure of the set of angles of rays landing on the main molecule.

How to compute the entropy

Let f be PCF.

Method 1 : construct Markov partition

You need to know the branch points.



Then the leading eigenvalue is $e^{h(f)}$.

Problem If I give you the external angle in M (e.g. $\theta = \frac{1}{5}$), it is hard to guess the position & number of branch points.

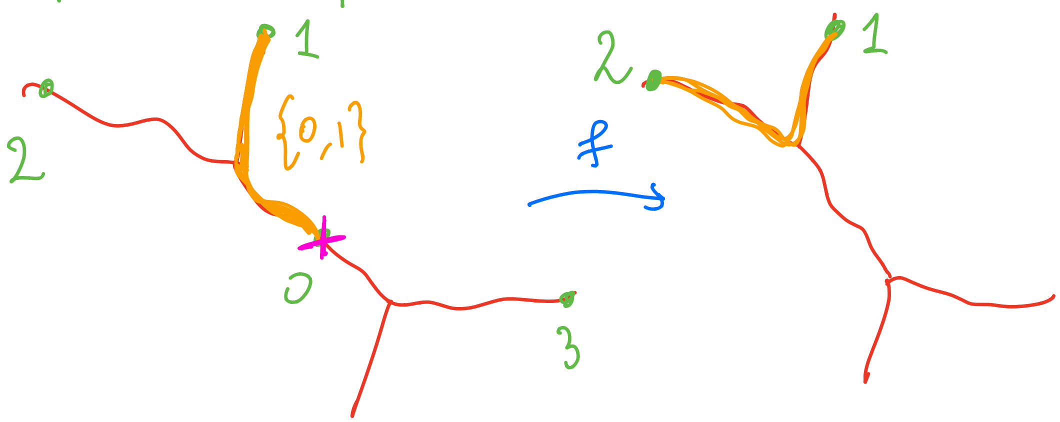
Method 2 (Thurston ; Tan Lei - Gao Yan)

Idea: consider all possible pairs between points in postcritical set

E.g.: $\mathcal{P} = \{0, 1, 2, 3\}$ then

$\text{Pairs} = \{\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

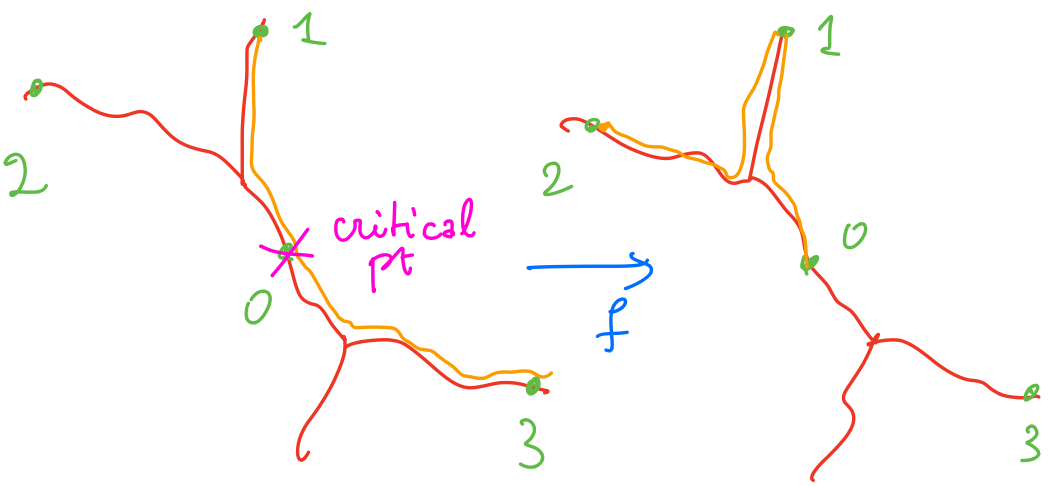
Each pair corresponds to an arc in H. tree



Then determine transitions between segments.

E.g.: $\{0, 1\} \longrightarrow \{1, 2\}$

$\{1, 3\} \longrightarrow \{0, 1\} \cup \{1, 2\}$



folding around critical point

Def.: an arc is non-separated if it does not contain the critical pt in its interior, and separated if it does.

Rule to Construct Transitions

1. If $\{i,j\}$ is non-separated, then

$$\{i,j\} \longrightarrow \{i+1, j+1\}$$

2. If $\{i,j\}$ is separated, then

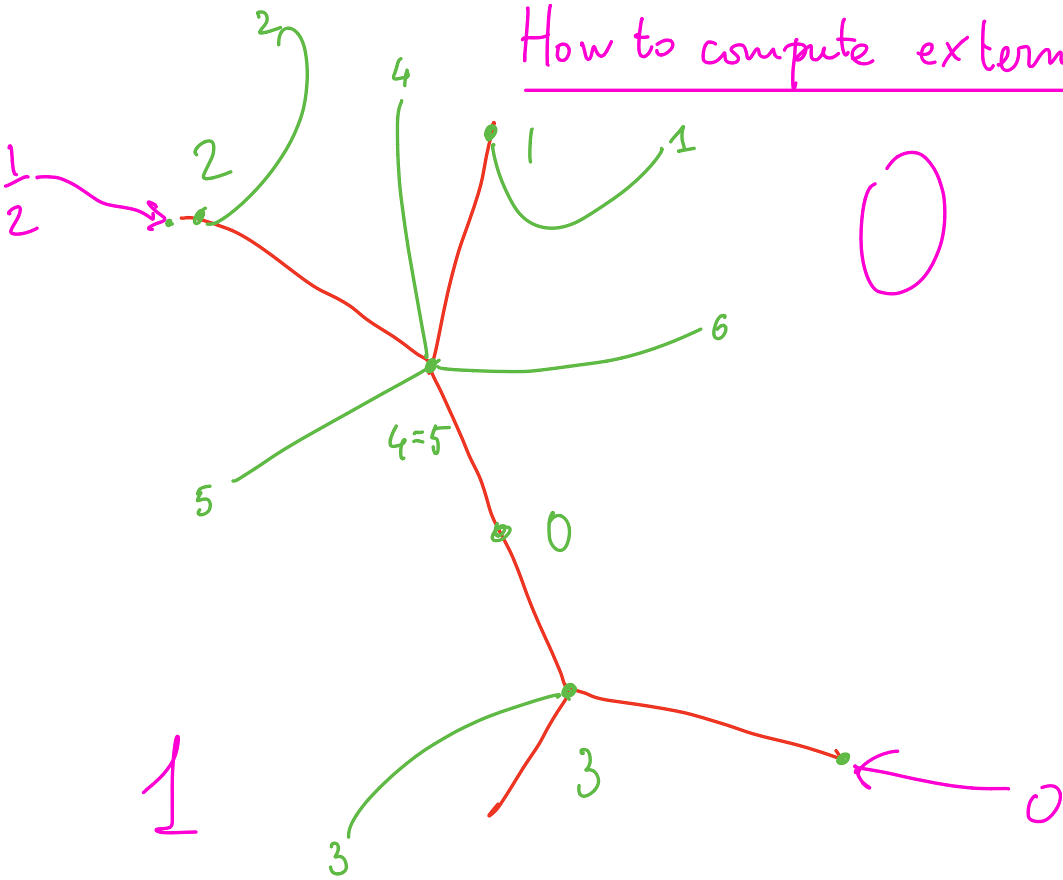
$$\{i,j\} \longrightarrow \{1, i+1\} \cup \{1, j+1\}$$

$$\| $$

$$\{i,0\} \cup \{0,j\}$$

Note: these rules are universal for all quadratic polynomials,

How to compute external angles?



$$.001\overline{010} = \frac{1}{8} + \frac{1}{32}\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots\right)$$

2 4 8 16 32 64

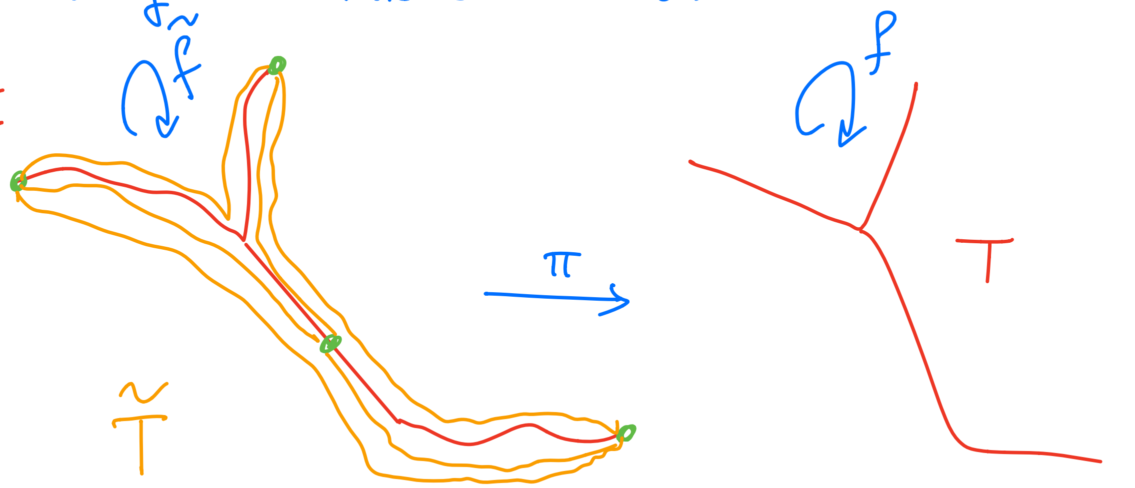
$$= \frac{1}{8} + \frac{1}{32}\,\frac{1}{1-\frac{1}{8}} =$$

$$= \frac{1}{8} + \frac{1}{\underset{4}{\cancel{32}}}\cdot\frac{\cancel{8}}{7} = \frac{1}{8} + \frac{1}{28} = \frac{7+2}{56}$$

$$= \frac{9}{56}$$

Thm If f is PCF, then the core entropy of f is the log of the leading eigenvalue of the matrix coding the transitions between pairs of postcritical arcs.

Pf



The log of the leading eig. yields the entropy of the map $\tilde{f}$ on the disjoint union $\tilde{T}$ of all postcritical arcs.

Hence the collapsing map π provides a semiconjugacy $\pi \circ \tilde{f} = f \circ \pi$

In the Misiurewicz case, π is finite-to-one. Hence

$$\log \lambda = h_{top}(\tilde{f}|_{\tilde{T}}) = h_{top}(f|_T).$$

$\boxed{Q}$ How does $h(\theta)$ change with θ ?

Theorem (T)

The core entropy function $h : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ extends to a continuous function
$$\tilde{h} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}.$$

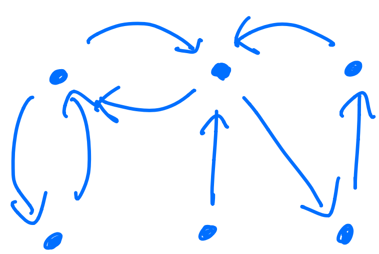
Note: if θ defines a topologically finite polynomial, then
$$\tilde{h}(\theta) = h_{top}(f_\theta |_{T_\theta}).$$

Problems

1. How to adapt Thurston's algo to irrational angles θ ?
2. How to control matrix when you change θ (even for $\theta \in \mathbb{Q}$?).

A bit of graph theory

Let Γ be a directed graph.



2 2-cycles
1 3-cycles
1 (2+3)-cycles

Characteristic Poly

$$Q(t) = \det(A - tI)$$

where A is adjacency matrix.

Thm If $P(t) := \det(I - tA)$, then

$$P(t) = \sum_{\substack{\gamma \text{ simple} \\ \text{multicycle}}} (-1)^{C(\gamma)} t^{\ell(\gamma)} \qquad (*)$$

A simple multicycle is a (vertex)-disjoint union of simple cycles.

$\ell(\gamma)$ = length of multicycle
$C(\gamma)$ = # connected components.

E.g.:

$$P(t) = 1 - 2t^2 - t^3 + t^5$$

Infinite graphs

We say countable, directed graph Γ has **bounded cycles** if

① The number of outgoing edges from a vertex is uniformly bounded

② For each n, there are only finitely many cycles of length n.

Rmk.: For a graph with bounded cycles, $P(t)$ is still defined as a formal power series.

Def.:

growth rate

$$r(\Gamma) := \limsup_{n\to\infty} \sqrt[n]{\mathcal{C}(\Gamma, n)}$$

$\mathcal{C}(\Gamma, n)$ is # of closed paths of length n.

$$\sigma(\Gamma) := \limsup_{n\to\infty} \sqrt[n]{\mathcal{S}(\Gamma, n)}$$

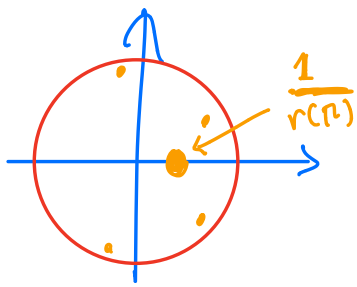
$\mathcal{S}(\Gamma, n)$ is # simple multicycles of length n

$(\sigma(\Gamma) \leq r(\Gamma))$

Thm (T) If Γ has bounded cycles and $\sigma(\Gamma) \leq 1$, then

$$\circledast \quad P(t) = \sum_{\gamma \text{ s.m.c.}} (-1)^{c(\gamma)} t^{\ell(\gamma)}$$

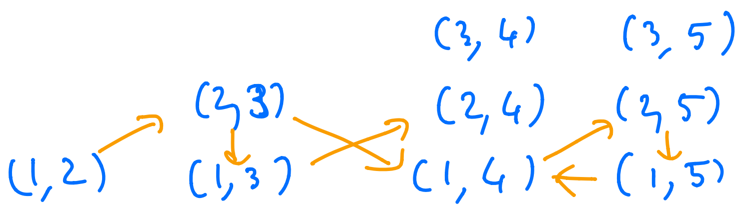
is a well-defined holomorphic function in $\{|t|<1\}$. Moreover, the smallest real root of $P(t)$ equals $\dfrac{1}{r(\Gamma)}$.



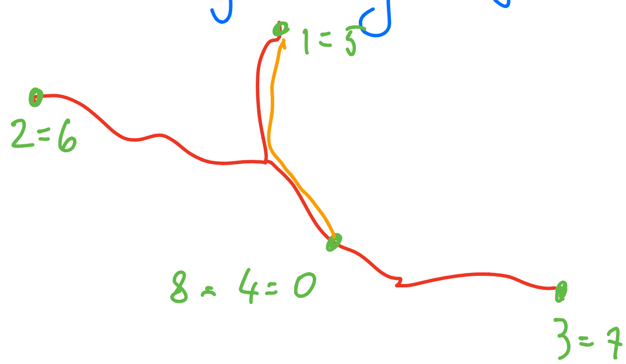
Idea of continuity

Define infinite graph with vertices

$$\Sigma = \{ (i,j) : 1 \le i \le j \}$$



Each (i,j) corresponds to the arc in H. tree joining $f^i(c)$ and $f^j(c)$



Define transition rules $(i,j) \to (i_1, j_1)$ depending on whether the pair (i,j) is separated or not.

Construct for each angle $\theta \in \mathbb{Q}/\mathbb{Z}$, a power series

$$P_\theta(t) \text{ using } *$$

Fundamental property:

if $\theta_n \to \theta$, then the labels of pairs (i,j) eventually stabilize.

Hence also

$$P_{\theta_n}(t) \longrightarrow P_\theta(t)$$

Hence, $r(\Gamma_{\theta_n}) \longrightarrow r(\Gamma_\theta)$ by Rouché's theorem.